

# Non-modal analysis of the diocotron instability. Plane geometry

V.V. Mikhailenko\* and H.J. Lee

*Pusan National University, 30 Jangjeon-dong,  
Guemjeong-gu, Pusan 609-735, S. Korea.*

V.S. Mikhailenko

*V.N. Karazin Kharkov National University, 61108 Kharkov, Ukraine.*

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## Abstract

The comprehensive investigation of the temporal evolution of the diocotron instability of the plane electron strip on the linear stage of its development is performed. By using the Kelvin's method of the shearing modes we elucidate the role of the initial perturbations of the electron density, which is connected with problem of the continuous spectrum. The linear non-modal evolution process, detected by the solution of the initial value problem, leads towards convergence to the phase-locking configuration of the mutually growing normal modes.

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\* Pusan National University, 30 Jangjeon-dong, Guemjeong-gu, Pusan 609-735, S. Korea.; E-mail: vladimir@pusan.ac.kr

## I. INTRODUCTION

The diocotron instability is one of the most ubiquitous instabilities[1] in low-density non-neutral plasmas with shear in the flow velocity. The goal of this study is to gain further insight into the physics of the development of the diocotron instability, assessing the role of the continuous spectrum [2, 3] and the recognition the dynamics of the relative phase of the diocotron waves on the linear stage of the instability development. These two problems are of paramount importance in understanding the temporal evolution of the initial perturbations into patterns peculiar to diocotron instability and has the major practical motivations. The solution of both these problems is amenable to examination using an initial value formulation of the linearised diocotron instability problem.

Since the first papers[4] devoted to the theory of the diocotron instability, the complete analogy of the stability problem of two-dimensional cold low-density electron beams in crossed external magnetic and self electric fields, and the problem of stability of the incompressible inviscid shear flows was recognized. Last time, valuable progress was obtained in the investigations of the stability of shear flows in a geophysical context[5–7]. A future common to these investigations is the recognition that a key element in the understanding the dynamics of the instability development, formation of the edges pattern and development of the vortex structures is the relative phase of the edge waves. In this paper we adopt this view point to examine inter-level interaction of edge diocotron waves in what is probably the simplest non-trivial case – we consider the plane model of the unbounded along the coordinate  $y$  non-neutral electron flow, which is confined in the strip  $-a \leq x \leq a$  and moves with ExB velocity in homogeneous magnetic field  $\mathbf{B} = B\mathbf{e}_z$  and in own crossed electric field,  $\mathbf{E} = E(x)\mathbf{e}_x$ , of the electrons strip. Our analysis is grounded on the method of the shearing modes (or so-called non-modal approach), developed in first by Lord Kelvin[8] (see, also Refs.[9, 10]). In the next Section, we formulate the basic non-modal equations. In Sec.III, these equations are solved for the case, when the initial perturbations of the electron density may be ignored and give the modal theory of the diocotron instability. In Sec.IV, we consider the diocotron instability in terms of edge waves interactions. The non-modal analysis of the diocotron instability, is presented in Sec.V, in which the role of the initial perturbations of the electron density on the temporal evolution of the diocotron instability to the phase-locking configuration, discovered in Sec.IV, is analysed. A summary of the

work is given in Conclusions, Section VI.

## II. BASIC EQUATION

The basic equation in the theory of the diocotron instability in plane geometry is the equation for the perturbed electrostatic potential  $\phi$  [1]

$$\left( \frac{\partial}{\partial t} + V'_0 x \frac{\partial}{\partial y} \right) \nabla^2 \phi(x, y, t) = \frac{4\pi e c}{B_0} \frac{\partial \phi}{\partial y} \frac{dn_0}{dx}. \quad (1)$$

In this paper we consider the homogeneous basic density of electrons, for which velocity shearing rate is  $V'_{0y} = \omega_{pe}^2 / \omega_{ce}$ , where  $\omega_{pe}$  and  $\omega_{ce}$  are electron plasma frequency and electron cyclotron frequency, respectively. We consider tenuous electron layer satisfying  $\omega_{pe} \ll \omega_{ce}$ , confined in a strip with edges at  $x = \pm a$ , i.e.  $n_0(x) = n_0(\Theta(x+a) - \Theta(x-a))$ , where  $\Theta(x-a)$  is the unit-step Heaviside function (it is equal to zero for  $x < a$  and equal to unity for  $x \geq a$ ); thus,  $dn_0(x)/dx = n_0(\delta(x+a) - \delta(x-a))$ . In this case Eq.(1) is written in the form

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + V'_0 x \frac{\partial}{\partial y} \right) \nabla^2 \phi(x, y, t) \\ &= V'_0 (\delta(x+a) - \delta(x-a)) \frac{\partial \phi}{\partial y}. \end{aligned} \quad (2)$$

Now we define boundary conditions for potential  $\phi(x, y, t)$ . We suppose, that potential is continuous across the surfaces  $x = \pm a$ , i.e.  $\phi(x = \pm a - \epsilon, y, t) = \phi(x = \pm a + \epsilon, y, t)$  with  $\epsilon \rightarrow 0$ . The conditions on the jump of the  $d\phi/dx$  at  $x = \pm a$  are determined by the integration[4] of Eq.(1) for the short distances  $\pm\epsilon \rightarrow 0$  across the both surfaces  $x = \pm a$ ,

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + V'_0 a \frac{\partial}{\partial y} \right) \left[ \frac{\partial \phi}{\partial x} \Big|_{x=a+\epsilon} - \frac{\partial \phi}{\partial x} \Big|_{x=a-\epsilon} \right] \\ &= -V'_0 \frac{\partial}{\partial y} \phi(x = a, y, t) \end{aligned} \quad (3)$$

$$\begin{aligned} & \left( \frac{\partial}{\partial t} - V'_0 a \frac{\partial}{\partial y} \right) \left[ \frac{\partial \phi}{\partial x} \Big|_{x=-a+\epsilon} - \frac{\partial \phi}{\partial x} \Big|_{x=-a-\epsilon} \right] \\ &= V'_0 \frac{\partial}{\partial y} \phi(x = -a, y, t) \end{aligned} \quad (4)$$

Also we require, that potential decays in vacuum region, i.e.  $\phi(x = \pm\infty, y, t) = 0$ .

We describe two areas: the electron layer,  $-a \leq x \leq a$ , and vacuum in the rest of space. Eq.(2) in the vacuum have a form

$$\frac{\partial}{\partial t} \nabla^2 \phi = 0. \quad (5)$$

The Fourier transformed over  $y$  solutions  $\phi(x, l, t) = \int \phi(x, y, t) \exp(-ily) dy$  of Eq.(5) are

$$\phi(x, l, t) = C_1(l, t) e^{-lx} \quad \text{for} \quad x > a, \quad (6)$$

$$\phi(x, l, t) = C_2(l, t) e^{lx} \quad \text{for} \quad x < -a. \quad (7)$$

In electron layer, the right hand side side of Eq. (2) is equal to zero, except the edges at  $x = \pm a$ , i.e.

$$\left( \frac{\partial}{\partial t} + V_0' x \frac{\partial}{\partial y} \right) \nabla^2 \phi = 0. \quad (8)$$

In the usually applied normal-mode (modal) approach, the solution to Eq.(8) is sought in the form  $\phi = \phi(x) \exp(-i\omega t + ik_y y)$ , for which Eq.(8) transforms to the form[4, 11]

$$(\omega - k_y V_0' x) \nabla^2 \phi = 0. \quad (9)$$

It is assumed, that  $\omega - k_y V_0' x \neq 0$ , and equation  $\nabla^2 \phi = 0$  is solved[4, 11]. The proper accounting for the resonance  $\omega - k_y V_0' x = 0$  is attained by solving the initial value problem to Eq.(9)[2, 3]. The principal result of the solution of the initial value problem by using the Laplace transform in time[2, 3] was that, in addition to the discrete eigenvalues linked to the normal modes, there exists a continuous spectrum of eigenvalues. The application of the Laplace transform to the solution of that problem[2, 3] made the calculation complicate and low effective - the explicit results for the temporal evolution of the potential were obtained only for the asymptotically large time.

Here we use other approach, which gives easy and transparent treating of the problem considered without the application of the spectral transforms in time. It is known, that for homogeneous shear flows, the solution of the initial value problem is greatly facilitated by a transformation of coordinates  $x, y$  to a new set of coordinate frame  $\xi, \eta$ , that is sheared with the mean flow. Such a transformation was proposed by Lord Kelvin[8] and is given by (see also Refs.[9, 10])

$$x = \xi, \quad y = \eta + V_0' \xi t, \quad t = t. \quad (10)$$

In the convected coordinate frame, Eq.(8) is spatially homogeneous and the inhomogeneity introduced by flow velocity in Eq.(8) is transformed to a temporal inhomogeneity,

$$\frac{\partial}{\partial t} \left[ \frac{\partial^2 \phi}{\partial \xi^2} - 2V_0' t \frac{\partial^2 \phi}{\partial \xi \partial \eta} + \left( 1 + (V_0' t)^2 \right) \frac{\partial^2 \phi}{\partial \eta^2} \right] = 0. \quad (11)$$

The integration of Eq.(11) over time,

$$\begin{aligned} \frac{\partial^2 \phi}{\partial \xi^2} - 2V_0' t \frac{\partial^2 \phi}{\partial \xi \partial \eta} + \left(1 + (V_0' t)^2\right) \frac{\partial^2 \phi}{\partial \eta^2} \\ = 4\pi e n_1(t=0, \xi, \eta), \end{aligned} \quad (12)$$

brings into the considered problem the initial perturbation of the density of electrons in layer,  $n_1(t=0, \xi, \eta)$ . After Fourier transforming over  $\eta$  of Eq. (12), we have

$$\begin{aligned} \frac{\partial^2 \phi}{\partial \xi^2} - 2iV_0' l t \frac{\partial \phi}{\partial \xi} - l^2 \left(1 + (V_0' t)^2\right) \phi \\ = 4\pi e n_1(t=0, \xi, l), \end{aligned} \quad (13)$$

The solution to Eq.(13) for potential  $\phi_1$  straightforwardly gives the initial value problem solution for the separate spatial Fourier harmonic for any desired time (without usual application[2] of the Laplace transform with time). That solution,

$$\begin{aligned} \phi(\xi, l, t) = \left( C_3(l, t) + \frac{2\pi e}{l} \int_{-a}^{\xi} d\xi_1 n_1(\xi_1, l) e^{-k_1 \xi_1} \right) e^{k_1 \xi} \\ + \left( C_4(l, t) + \frac{2\pi e}{l} \int_{\xi}^a d\xi_1 n_1(\xi_1, l) e^{-k_2 \xi_1} \right) e^{k_2 \xi}, \end{aligned} \quad (14)$$

has an obvious non-modal form with non-separable dependences on time and coordinate in exponentials, resulted from the time-dependent  $k_{1,2} = \pm l + iV_0' l t$ . The temporal evolution of the potential appears to be the strictly non-modal process.

### III. MODAL DIOCOTRON INSTABILITY

If we suppose that any initial perturbation in layer are absent, i.e.  $n_1(\xi_1, l) = 0$ , then we obtain the solution, which describes only the surface waves, which form the discrete spectrum of perturbations

$$\phi(\xi, l) = C_3(l, t) e^{k_1 \xi} + C_4(l, t) e^{k_2 \xi}. \quad (15)$$

The condition of the continuity of the perturbed potential on the boundaries ( $x = \xi = \pm a$ ) couples  $C_1(l, t)$ ,  $C_2(l, t)$  and  $C_3(l, t)$ ,  $C_4(l, t)$ , and gives the following presentation for the potential:

$$\phi(\xi, l) = \left( C_3(l, t) + C_4(l, t) e^{2al} \right) e^{l\xi - iV_0' alt}, \xi < -a$$

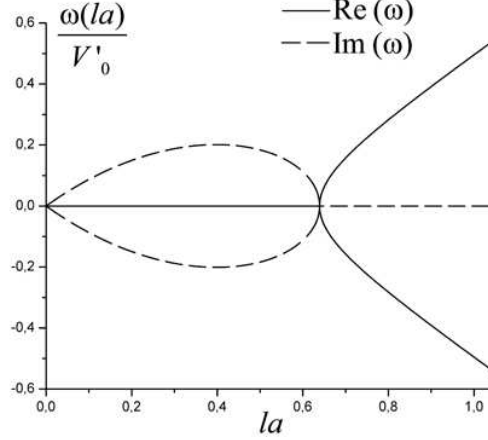


FIG. 1. The dispersion relation of the normal modes.

$$\begin{aligned}\phi(\xi, l) &= \left( C_3(l, t) e^{l\xi} + C_4(l, t) e^{-l\xi} \right) e^{iV'_0 \xi l t}, \quad -a < \xi < a, \\ \phi(\xi, l) &= \left( C_3(l, t) e^{2al} + C_4(l, t) \right) e^{-l\xi + iV'_0 alt}, \quad \xi > a.\end{aligned}\tag{16}$$

Now we apply boundary conditions (3), (4) to (16), and obtain the system of equations for  $C_3(l, t)$  and  $C_4(l, t)$ , i.e.

$$\frac{\partial C_3}{\partial t} = i \frac{V'_0}{2} (1 - 2la) C_3 + i \frac{V'_0}{2} e^{-2al} C_4,\tag{17}$$

$$\frac{\partial C_3}{\partial t} = -i \frac{V'_0}{2} e^{-2al} C_3 - i \frac{V'_0}{2} (1 - 2la) C_4.\tag{18}$$

The solution to Eqs. (17), (18) gives the relation

$$\omega = \pm \frac{V'_0}{2} \sqrt{(1 - 2la)^2 - e^{-4al}}.\tag{19}$$

which is the known dispersion equation for diocotron oscillations in plane charge sheet[11]. The dispersion equation (19) is illustrated in Fig.1. as a function of the parameter  $al$ . In the case  $e^{-4al} > (1 - 2la)^2$ , we have imaginary frequency which defines the growth rate of the diocotron instability. In the case, when boundaries of electron layer are so far from each other that  $e^{-4al} < (1 - 2la)^2$ , we have two not interacting stable waves.

#### IV. MODAL DIOCOTRON INSTABILITY INTERPRETED IN TERMS OF EDGE WAVES INTERACTION

The result of the above analysis is the known dispersion equation (19). This equation, however, does not give an overview of how different initial wave structures will evolve in

time. Writing the functions  $C_3(l, t)$  and  $C_4(l, t)$  in complex form[5],

$$C_3(l, t) = Q_3(l, t) e^{i\epsilon_3(l, t)}, \quad (20)$$

$$C_4(l, t) = Q_4(l, t) e^{i\epsilon_4(l, t)}, \quad (21)$$

the edge perturbation of the potential can be regarded as two edge waves with amplitudes  $Q_3(l, t)$  and  $Q_4(l, t)$  and phases  $\epsilon_3(l, t)$  and  $\epsilon_4(l, t)$ . By substituting Eqs.(20), (21) into Eqs.(17), (18) and separating the real and imaginary parts at  $x = \pm a$ , we obtain, that amplitudes  $Q_3(l, t)$  and  $Q_4(l, t)$ , and the relative phase  $\epsilon = \epsilon_3 - \epsilon_4$  of the edge diocotron waves evolve according to equations

$$\frac{dQ_3}{dt} = \frac{V'_0}{2} e^{-2al} Q_4 \sin \epsilon, \quad (22)$$

$$\frac{dQ_4}{dt} = \frac{V'_0}{2} e^{-2al} Q_3 \sin \epsilon, \quad (23)$$

$$\frac{d\epsilon}{dt} = \Gamma (\cos \epsilon + b(t)), \quad (24)$$

where

$$\Gamma = \frac{V'_0}{2} e^{-2al} \left( \frac{Q_3}{Q_4} + \frac{Q_4}{Q_3} \right), \quad (25)$$

and

$$b(t) = \frac{2(1 - 2la) e^{2al}}{\left( \frac{Q_3}{Q_4} + \frac{Q_4}{Q_3} \right)}, \quad (26)$$

It follows from (19), that at the condition of the diocotron instability development,  $(1 - 2la) e^{2al} < 1$ . Also, it is easily obtained, that

$$\frac{Q_3}{Q_4} + \frac{Q_4}{Q_3} \geq 2 \quad (27)$$

(the equality sign appears when  $Q_3 = Q_4$ ). From Eqs.(22), (23) one can obtain the integral,

$$Q_3^2 - Q_4^2 = C. \quad (28)$$

Therefore, due to the exponential growth of amplitudes  $Q_3, Q_4$  with time from infinitesimal beginnings, the amplitudes become almost equal,  $Q_3^2 = Q_4^2 + C \approx Q_4^2 \gg C$ , and  $b(t)$  approaches the value  $b_0 = (1 - 2la) e^{2al}$ . Therefore, at the condition, under which the diocotron instability develops, we have  $b(t) < 1$ , and therefore, the stationary (or fixed) points[12] of the equation (24), where  $d\epsilon/dt = 0$ , exist and are determined by the equation

$\cos \epsilon + b_0 = 0$ . The solutions of this equation are two sets of stationary points: stable (or attractors) at

$$\epsilon_k = (\pi - \cos^{-1} b_0) + 2k\pi, \quad (29)$$

and unstable at

$$\epsilon_k = -(\pi - \cos^{-1} b_0) + 2k\pi. \quad (30)$$

Note, that solution of the equation  $d\epsilon/dt = \cos \epsilon + b_0$  with initial condition  $\epsilon = \epsilon_0$  at  $t = t_0 = 0$ , has a simple form

$$\tan \frac{\epsilon}{2} = -\sqrt{\frac{1+b_0}{1-b_0}} \left( \frac{1 + Ae^{t\sqrt{1-b_0^2}}}{1 - Ae^{t\sqrt{1-b_0^2}}} \right) \quad (31)$$

where

$$A = \frac{(1-b_0) \tan \frac{\epsilon_0}{2} + \sqrt{1-b_0^2}}{(1-b_0) \tan \frac{\epsilon_0}{2} - \sqrt{1-b_0^2}} \quad (32)$$

As it follows from Eq.(31), the initial perturbations with arbitrary value of the initial phase of each wave, will evolve with time to the ultimate value of relative phase,

$$\tan \frac{\epsilon}{2} = \sqrt{\frac{1+b_0}{1-b_0}} \quad (33)$$

which does not depend on the initial data and is the same as established by Eq.(29) (see Fig.2). For  $al = 0.4$ , Eq.(33) gives  $\epsilon \approx 110^\circ$  (see Fig.3, where that phase locking configuration is presented). This solution of the initial value problem reveals the initial stage of the instability development as a process of the formation of the phase locked configuration. As it follows from Eqs.(22)–(24), the time of the developing of the phase locking configuration is comparable with of the inverse growth rate time,  $t \gtrsim \Gamma^{-1} \simeq (V_0')^{-1}$  of the diocotron instability. Two edge diocotron waves, embedded within spatially distinct regions of oppositely directed density gradients interact such, that the wave trains transit to a phase-locked state of mutual growth.

## V. NON-MODAL ANALYSIS OF THE DIOCOTRON INSTABILITY.

In the considered above idealized case, when the initial perturbation  $n_1(t=0, \xi, \eta)$  of the electron density is ignored, Eqs. (17), (18) resulted from the boundary conditions at  $\xi = \pm a$ ,



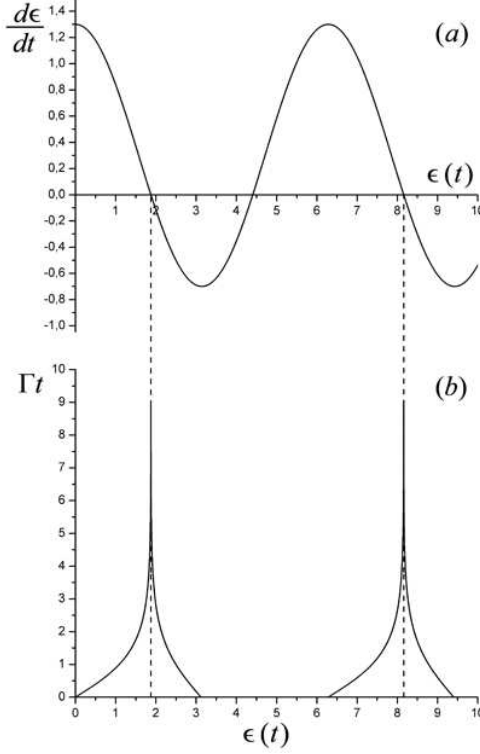


FIG. 2. (a)Phase portrait for Eq.(24); (b) The evolution of the relative phase  $\epsilon$  with time.

in which non-modal non-separable structure as  $\xi t$  of the spatio-temporal dependence of the solution (14) receives ordinary modal, as  $at$ , form. As a result, we obtain modal theory of the diocotron instability. The accounting for the initial perturbation of the electron density excludes such simplifications and complete non-modal analysis becomes necessary. In this section, we solve for solution (14) the boundary value problem, determined by the condition of the continuity of the potential  $\phi_1$  and by Eqs.(3), (4). The continuity of the potential gives the following presentation of the solutions (6), (7) through the functions  $C_3(l, t)$  and  $C_4(l, t)$  of the solution (14) in different regions of the space. We obtain for  $\xi < -a$

$$\begin{aligned} \phi(\xi, l, t) = & \left( C_3(l, t) e^{-k_1 a} + C_4(l, t) e^{-k_2 a} \right. \\ & \left. + \frac{2\pi e}{l} \int_{-a}^a d\xi_1 n_1(\xi_1, l) e^{-k_1(\xi_1 + a)} \right) e^{l(\xi + a)}, \end{aligned} \quad (34)$$

and for  $\xi > a$

$$\phi(\xi, l, t) = \left( C_3(l, t) e^{k_1 a} + C_4(l, t) e^{k_2 a} \right)$$

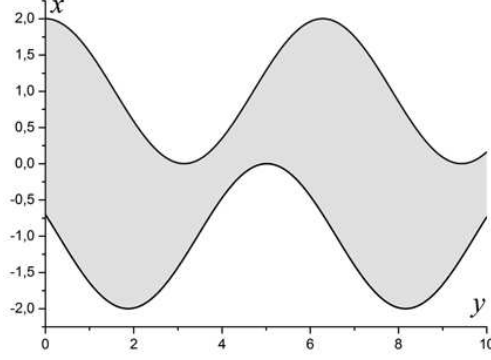


FIG. 3. The phase-locking configuration with relative phase  $\epsilon \approx 110^\circ$ , obtained from Eq.(30) for  $al = 0.4$ .

$$+ \frac{2\pi e}{l} \int_{-a}^a d\xi_1 n_1(\xi_1, l) e^{-k_1(\xi_1 - a)} \Big) e^{-l(\xi - a)}, \quad (35)$$

The application of the conditions (3), (4) to solutions (14), (34), and (35) gives the inhomogeneous equations for  $C_3(l, t)$  and  $C_4(l, t)$ ,

$$\frac{\partial C_3}{\partial t} = i \frac{V'_0}{2} (1 - 2la) C_3(l, t) + i \frac{V'_0}{2} e^{-2al} C_4(l, t) + f_1(l, t), \quad (36)$$

$$\frac{\partial C_4}{\partial t} = -i \frac{V'_0}{2} e^{-2al} C_3(l, t) - i \frac{V'_0}{2} (1 - 2la) C_4(l, t) + f_2(l, t). \quad (37)$$

where

$$f_1(l, t) = i\pi e \frac{V'_0}{l} \int_{-a}^a d\xi_1 n_1(\xi_1, l) e^{-(l+iV'_0 t)\xi_1} (1 + 2l(\xi_1 - a)), \quad (38)$$

$$f_2(l, t) = -i\pi e \frac{V'_0}{l} \int_{-a}^a d\xi_1 n_1(\xi_1, l) e^{(l-iV'_0 t)\xi_1} (1 - 2l(\xi_1 + a)). \quad (39)$$

The solution to system (36)-(39) for  $C_3(l, t)$  and  $C_4(l, t)$  is obtained straightforwardly and is given by

$$C_3(l, t) = c_1 e^{\gamma t} + c_2 e^{-\gamma t} + \hat{C}_3(l, t), \quad (40)$$

and

$$C_4(l, t) = c_1 \alpha_1 e^{\gamma t} + c_2 \alpha_2 e^{-\gamma t} + \hat{C}_4(l, t), \quad (41)$$

where

$$\begin{aligned} \hat{C}_3(l, t) &= \frac{\pi e (V'_0)^2}{4l\gamma} e^{-2la} \sum_{m=-\infty}^{\infty} n_1(m, l) e^{im\pi} \\ &\times \left( e^{\gamma t} (-\alpha_2 I_1 + I_2) + e^{-\gamma t} (-I_3 + \alpha_1 I_4) \right), \end{aligned} \quad (42)$$

$$\begin{aligned} \hat{C}_4(l, t) &= \frac{\pi e (V'_0)^2}{4l\gamma} e^{-2la} \sum_{m=-\infty}^{\infty} n_1(m, l) e^{im\pi} \\ &\times \left( \alpha_1 e^{\gamma t} (-\alpha_2 I_1 + I_2) + \alpha_2 e^{-\gamma t} (\alpha_1 I_3 - I_4) \right). \end{aligned} \quad (43)$$

In Eqs.(42), (43)

$$\begin{aligned} I_{1,3} &= \int_0^t dt_1 \frac{e^{-k_1 a \mp \gamma t_1}}{i \frac{m\pi}{a} - k_1} \left( 1 - \frac{2l}{i \frac{m\pi}{a} - k_1} \right) \\ &- \int_0^t dt_1 \frac{e^{k_1 a \mp \gamma t_1}}{i \frac{m\pi}{a} - k_1} \left( 1 - 4al - \frac{2l}{i \frac{m\pi}{a} - k_1} \right), \end{aligned} \quad (44)$$

$$\begin{aligned} I_{2,4} &= \int_0^t dt_1 \frac{e^{k_2 a \mp \gamma t_1}}{i \frac{m\pi}{a} - k_2} \left( 1 - \frac{2l}{i \frac{m\pi}{a} - k_2} \right) \\ &+ \int_0^t dt_1 \frac{e^{-k_2 a \mp \gamma t_1}}{i \frac{m\pi}{a} - k_2} \left( -1 + 4al - \frac{2l}{i \frac{m\pi}{a} - k_2} \right), \end{aligned} \quad (45)$$

where  $k_{1,2} = \pm l + iV'_0 l t$ ,  $\gamma = i\omega$  is the growth rate, determined by Eq.(19),  $\alpha_{1,2} = -e^{2la} \left( (1 - 2la) \pm \frac{2i\gamma}{V'_0} \right)$ , and the presentation of the initial density perturbation in a form of Fourier series,

$$n_1(\xi, l) = \sum_{m=-\infty}^{\infty} n_1(m, l) e^{\frac{im\pi\xi}{a}},$$

was used. Eqs.(14), (34), (35), and (42)–(45) compose the complete explicit solution on the initial and boundary value problems for the separate Fourier harmonic  $\phi(\xi, l, t)$ . That solution is valid for all times, at which linear theory of the diocotron instability is applicable. As it follows from Eqs.(44), (45), at time  $t < t_* = \pi m / la V'_0$  the denominators in integrands of  $I_{1,3}$  and  $I_{2,4}$  decrease with time and approach their minimal value  $\pm l$  at time  $t = t_*$ , which is of the order of a few inverse growth rate times. Only after time  $t_*$ , denominators grow with time as  $t$ , leading to the decay of the potential with time as  $(V'_0 t)^{-1}$ . It is important to note, that only that last stage of the potential evolution in time is amenable analytically by using inverse Laplace transform[2, 3], temporal growth of the potential at time  $t < t_*$ , known as the Orr mechanism[7, 13], was completely overlooked.

The solution for  $\phi(\xi, l, t)$  for time  $t > t_*$  is easily obtained by integration of  $I_{1,3}$  and  $I_{2,4}$  by parts and may be presented in the form

$$\phi(\xi, l, t) = \phi_{(0)}(\xi, l, t) + \hat{\phi}(\xi, l, t), \quad (46)$$

where

$$\begin{aligned} \phi_{(0)}(\xi, l, t) = & c_1 (e^{\gamma t + k_1 \xi} + \alpha_1 e^{\gamma t + k_2 \xi}) \\ & + c_2 (e^{-\gamma t + k_1 \xi} + \alpha_2 e^{-\gamma t + k_2 \xi}). \end{aligned} \quad (47)$$

The constants  $c_1$  and  $c_2$  are determined by the initial perturbations of the electrostatic potential (or electron density) on the boundary surfaces at  $x = \pm a$ . The potential  $\hat{\phi}$  is equal to

$$\hat{\phi}(\xi, l, t) = \hat{\phi}_1(\xi, l, t) + \hat{\phi}_2(\xi, l, t) + \hat{\phi}_3(\xi, l, t), \quad (48)$$

where

$$\begin{aligned} \hat{\phi}_1(\xi, l, t) = & \hat{C}_3(l, t) e^{k_1 \xi} \\ = & -\frac{\pi e (V'_0)^2}{4\gamma l} e^{-2la} \sum_{m=-\infty}^{\infty} n_1(m, l) e^{im\pi} \\ \times & \left\{ \frac{1}{k_1 - \frac{i\pi m}{a}} \left[ e^{k_1(\xi-a)} \left( \frac{\alpha_2}{\gamma + ilaV'_0} + \frac{\alpha_1}{\gamma - ilaV'_0} \right) \right. \right. \\ & \left. \left. - e^{k_1(\xi+a)} (1 - 4la) \left( \frac{\alpha_2}{\gamma - ilaV'_0} + \frac{\alpha_1}{\gamma + ilaV'_0} \right) \right] \right. \\ & \left. + \frac{2\gamma}{(\gamma^2 + (V'_0 la)^2) \left( \frac{i\pi m}{a} - k_2 \right)} \right. \\ & \left. \times (e^{k_1(\xi+a)} e^{-2la} - (1 - 4a) e^{k_1(\xi-a)} e^{2la}) \right\}, \end{aligned} \quad (49)$$

$$\begin{aligned} \hat{\phi}_2(\xi, l, t) = & \hat{C}_4(l, t) e^{k_2 \xi} \\ = & -\frac{\pi e (V'_0)^2}{4\gamma l} e^{-2la} \sum_{m=-\infty}^{\infty} n_1(m, l) e^{im\pi} \\ \times & \left\{ \frac{e^{k_2(\xi+a)}}{\left( \frac{i\pi m}{a} - k_2 \right)} \left( \frac{\alpha_1}{\gamma - ilaV'_0} + \frac{\alpha_2}{\gamma + ilaV'_0} \right) \right. \\ & - \frac{(1 - 4la) e^{k_2(\xi-a)}}{\left( \frac{i\pi m}{a} - k_2 \right)} \left( \frac{\alpha_2}{\gamma + ilaV'_0} + \frac{\alpha_1}{\gamma - ilaV'_0} \right) \\ & \left. + \frac{2\gamma \alpha_1 \alpha_2}{(\gamma^2 + (V'_0 la)^2) \left( k_1 - \frac{i\pi m}{a} \right)} \right. \\ & \left. \times (e^{k_2(\xi-a)} e^{-2la} - (1 - 4a) e^{k_2(\xi+a)} e^{2la}) \right\}, \end{aligned} \quad (50)$$

and  $\hat{\phi}_3$  is formed by the initial perturbations in Eq.(14),

$$\begin{aligned} \hat{\phi}_3(\xi, l, t) = & \frac{2\pi e}{l} \sum_{m=-\infty}^{\infty} n_1(m, l) \\ & \times \left[ \frac{\left( e^{i\pi m} e^{k_1(\xi+a)} - e^{\frac{i\pi m \xi}{a}} \right)}{\left( k_1 - \frac{i\pi m}{a} \right)} + \frac{\left( e^{i\pi m} e^{k_2(\xi-a)} - e^{\frac{i\pi m \xi}{a}} \right)}{\left( \frac{i\pi m}{a} - k_2 \right)} \right]. \end{aligned} \quad (51)$$

The presentation of the solution (46) in coordinates  $x, y, t$  is attained by performing the inverse Fourier transform for wavenumber  $l$  with changing the conjugate variable  $\eta$  using the transformations (10),

$$\begin{aligned} \phi(x, y, t) = & \frac{1}{2\pi} \int dl e^{ily - iV'_0 x t l} \left( \phi_{(0)}(x, l, t) \right. \\ & \left. + \hat{\phi}(x, l, t) \right) = \phi_{(0)}(x, y, t) + \hat{\phi}(x, y, t). \end{aligned} \quad (52)$$

It is interesting to note, that  $\phi_{(0)}(x, y, t)$  has a modal form,

$$\begin{aligned} \phi_{(0)}(x, y, t) = & \frac{1}{2\pi} \int dl e^{ily} \left( c_1(l) e^{\gamma t + lx} \right. \\ & \left. + c_2(l) e^{-\gamma t + lx} + c_1(l) \alpha_1 e^{\gamma t - lx} + c_2(l) \alpha_2 e^{-\gamma t - lx} \right), \end{aligned} \quad (53)$$

whereas  $\hat{\phi}(x, y, t)$  is non-modal in both sets of variables. The integrand, which determines  $\hat{\phi}(x, y, t)$  in Eq.(52), decays with time as  $(V'_0 t)^{-1}$  at times  $t > t_*$  and contains non-modal multiplier  $\exp(-iV'_0 x t l)$  (which cancelled in Eq.(53)). From the spectral point of view, solution (52) reveals the result, obtained in the asymptotic limit of large time by using Laplace transform[2, 3], that the spectrum of the diocotron instability contains discrete spectrum of two modes (solution  $\phi_{(0)}(x, y, t)$ ), and continuous spectrum, which is presented by  $\hat{\phi}(x, l, t)$ .

## VI. CONCLUSIONS

In this paper, we have performed the comprehensive investigation of the temporal evolution of the diocotron instability of the plane electron strip on the linear stage of its development. In the realistic laboratory flows, as well as in numerical simulations, perturbation growth occurs over a finite time interval due to disruption by instabilities and turbulence. It is therefore of interest to find all factors yielding the instability development over a specified time interval. We find, that normal mode theory, presented in Sec.III, is unable to explain

the physics of the evolutionary processes, which occur in the volume and on surfaces of the electron strip. The understanding of these inherently non-modal processes attains via solution of the initial value problem. By using the Kelvin's method of shearing modes we elucidate the role of the initial perturbations of the electron density, which is connected with problem of the continuous spectrum [2, 3]. Because the growth rate  $\gamma \lesssim 0.2V'_0$ , at time  $t \gtrsim \gamma^{-1} \sim 5(V'_0)^{-1}$ , effect of the non-modality in the solution for  $\phi(\xi, l, t)$  becomes important. We find, that linear non-modal evolution process, detected by the solution of the initial value problem, leads towards convergence to the phase-locking configuration of the growing normal modes, which is the precursor of the development of the multi vortex structure, observed in experiments and in numerical simulations.

It is important to note, that solutions (42)–(45), which are valid for any times, incorporate additional mechanism of the temporal growth of the initial disturbances, known as the Orr mechanism[7, 13] – solutions (42)–(45) experience the temporal growth at  $t \leq t_*$ , at which the denominators of Eqs.(44), (45) achieve their minimal values. Depending on the magnitude of the initial perturbations of the electron density, this effect may be comparable with modal growth, and it is crucial for the growth of the short along the shear flow perturbations, which are stable against the modal diocotron instability.

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- [1] R. Davidson, *An Introduction to the Physics of Non-neutral Plasmas*, Addison-Wesley Publishing Company. (2001)
  - [2] K.M. Case, Phys.Fluids, **3**, 143, (1960)
  - [3] R.J. Briggs, J.D. Daugherty, and R.H. Levy, Phys. Fluids 13, 421 (1970); doi: 10.1063/1.1692936
  - [4] R.H. Levy, Phys. Fluids, **8**, 1288, (1965)
  - [5] E. Heifetz, C.H. Bishop, P. Alpert, Quarterly Journal of the Royal Meteorological Society, **125**, 2835, (1999)
  - [6] E. Heifetz, C.H. Bishop, B.J. Hoskins, and J. Methven, Quarterly Journal of the Royal Meteorological Society, **130**, 211, (2004)
  - [7] N.A. Bakas, P.J. Ioannou, Physics of Fluids, **21**, 024102, (2009)
  - [8] Lord Kelvin, "Stability of fluid motion: rectilinear motion of viscous fluid between two parallel

- plates”, Philos. Mag. **24**, 188 (1887).
- [9] V.S. Mikhailenko, V.V. Mikhailenko, and K.N. Stepanov, Phys. Plasmas **7**, 94 (2000).
- [10] V.S. Mikhailenko, E.E. Scime, V.V. Mikhailenko, Physical Review, E**71**, 026306 (2005)
- [11] W. Knauer, Journal of Applied Physics, **37**, 602, (1966)
- [12] S.H. Strogatz, *Non-linear dynamics and chaos*. Perseus Book Publishing L.L.C., 1994, 498 p.
- [13] W.M.F. Orr, Proc.R.Ir.Acad., Sect. A 27, 9 (1907)